

# Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac statistical entropies in a $D$ -dimensional stationary axisymmetry space-time

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Statistical entropies of a general relativistic ideal gas obeying Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac statistics are calculated in a general axisymmetry space-time of arbitrary dimension. Analytical expressions for the thermodynamic potentials are presented, and their behaviors in the high or low temperature approximation are discussed. The entropy of a quantum field is shown to be proportional to the volume of optical space or that of the dragged optical space only in the high temperature approximation or in the zero mass case. In the case of a black hole, the entropy of a quantum field at the Hartle-Hawking temperature is proportional to the horizon "area" if and only if the horizon is located at the light velocity surface.

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## 1 Introduction

Recently many efforts have been focused on understanding the statistical origin of black hole entropy<sup>1</sup> and its interpretation<sup>2,3</sup>. One of them is the so-called "brick-wall model" introduced by t' Hooft<sup>4</sup>. By using this model, he first showed that the leading entropy of a quantum gas of scalar particles propagating outside the event horizon of the Schwarzschild black hole is proportional to its horizon area but diverges near the horizon. The divergences arise from the infinite one-particle number or state density of levels due to the presence of arbitrary high modes near the horizon. To remove this divergence, t' Hooft introduced a brick wall cutoff which is related to the horizon only.

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Many works on the application of the brick-wall model to various kinds of black hole have been done for scalar fields<sup>4–11</sup>. In four dimensional Schwarzschild black hole, besides t’ Hooft’s pioneer work<sup>3</sup>, the model has also been used to calculate the entropies for an idea gas obeying the usual three kinds of statistics<sup>5</sup>. They showed that the area law for the entropy of a quantum field is due to the quantum statistics. Ghosh and Mitra<sup>6</sup> applied it to the Schwarzschild dilatonic black hole. In the case of a four dimensional rotating black hole, Lee and Kim<sup>7,8</sup> demonstrated that the leading term of the entropy of a neutral scalar field is proportional to the area of the event horizon and diverges as the system approaches to the event horizon. Similar conclusion holds true also for the leading entropy of a complex scalar field at the Hartle-Hawking temperature of a Kerr-Newman black hole<sup>9</sup>. The reason of the divergences is attributed to the infinite number of state or the infinite volume of the phase space near the horizon. In other words, the origin of the divergence is that the density of states diverges at the horizon. In a  $2 + 1$  dimensional black hole, the entropy of a quantum scalar field has also been studied in Ref. 10 and 11.

It is well known that bosonic fields have a special class of superradiant mode solution<sup>12</sup>. Because its complexity, the contribution to the entropy from superradiant modes hadn’t been considered in Ref. 7-10. In the case of a Kerr black hole, it is claimed in Ref. 13 that the negative contribution to the entropy from superradiant modes is divergent in the leading order. However, Ho and Kang<sup>14</sup> pointed out that its entropy contribution is positive rather than negative and the previous error in Ref. 11 comes from the incorrect quantization of the superradiant modes.

It is generally believed now that the entropy of a quantum field in a black hole is proportional to the area of a black hole horizon but diverges due to the presence of the event horizon. However, Alwis and Ohta<sup>15</sup> demonstrated that the free energy and the entropy for a scalar field or a Dirac field in the high temperature approximation is proportional to the volume of optical space in a static space-time background. Thus the horizon area of a black hole must have a certain relation to the volume of its corresponding optical space.

Then the following questions arise: What relation between the entropy calculated by the brick wall model method and that by the optical method? Has any generality among the entropies of a quantum field in different dimensional space-times or in different kinds of black holes with the same dimensional number? Jing and Yan<sup>16</sup> had studied the entropy of a minimally coupled quantum scalar field in the four dimensional general non-extremal stationary axisymmetry black hole. However, for a general arbitrary dimensional space-time,

there has no similar work presently. To fill up this gap, we investigate quantum statistics of a relativistic idea gas obeying three Kinds of the usual statistics: Maxwell-Boltzmann (M-B), Bose-Einstein (B-E) or Fermi-Dirac (F-D) in an arbitrary  $D$ -dimensional space-time.

In this paper, we first calculate the state density of a relativistical idea gas obeying three kinds of the usual statistics. We assume that this state density is effective for these three kinds of the usual statistics. Then we do thermodynamics in a curved space-time like the usual non-relativistical ones<sup>17</sup> in the flat space-time. The paper is organized as follows: In Sec. 2, we give a general description of a  $D$ -dimensional stationary axisymmetry space-time including static space-times and the flat space-time as well as black hole solutions. Then in Sec. 3, we derive the constraint on momentum space and evaluate the state density of phase space for a given energy. Sec. 4 and 5 is devoted to calculating thermodynamical potentials in two cases: the angular velocity of a quantum field  $\Omega_0 = 0$  and  $\Omega_0 \neq 0$ , respectively. Analytical expressions of the thermodynamical potentials are given, their asymptotical behaviors in the high or low temperature are discussed. General discussions about the divergence of the entropy of the quantum fields in a black hole background are presented in Sec. 6. Some four dimensional space-times are considered as examples in Sec. 7. Finally, we present our conclusions and problems not being considered here.

## 2 Description of general space-time

To begin with, let us firstly give some general description of a  $D$ -dimensional stationary axisymmetry space-time. As specially important examples, the flat space-time or a static space-time as well as black hole solutions are included under our present consideration.

In general, the line element and electro-magnetic potential of a  $D$ -dimensional stationary axisymmetry space-time can be expressed in the following form:

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{ij}dx^i dx^j, \quad (1)$$

$$A = A_t dt + A_\varphi d\varphi + A_i dx^i, \quad i, j = 1, 2, \dots, D-2. \quad (2)$$

where the metric elements  $g_{tt}, g_{t\varphi}, g_{\varphi\varphi}$  and  $g_{ij}$  are functions of coordinates  $x = (x^1, x^2, \dots, x^{D-2})$  only. The metric signature is taken as  $(-, +, \dots, +)$  and its submatrix  $g_{ij}$  is assumed to be diagonal. This space-time (1) has two Killing vectors  $\partial_t$  and  $\partial_\varphi$  relating to two conserved quantities, energy and azimuthal angular momentum. As examples, it contains a large class of black hole solutions and non-black-hole solutions such as the flat space-time. Static black holes are included as special case when  $g_{t\varphi} = 0$ . The event horizon  $f(x) = 0$ , if it exists, is a null hypersurface determined by

$$g^{tt}\partial_t^2 f(x) + 2g^{t\varphi}\partial_t\partial_\varphi f(x) + g^{\varphi\varphi}\partial_\varphi^2 f(x) + g^{ij}\partial_i f(x)\partial_j f(x) = g^{ij}\partial_i f(x)\partial_j f(x) = 0. \quad (3)$$

Here the contravariant metric elements being

$$g^{tt} = g_{\varphi\varphi}/\mathcal{D}, g^{t\varphi} = -g_{t\varphi}/\mathcal{D}, g^{\varphi\varphi} = g_{tt}/\mathcal{D}, g^{ij} = 1/g_{ij}, \mathcal{D} = g_{tt}g_{\varphi\varphi} - (g_{t\varphi})^2, \quad (4)$$

provided the metric determinant is nonsingular, namely,  $g_D = \mathcal{D} \det g_{ij} = \mathcal{D}g_{D-2} \neq 0$ .

The condition that the nontrial null vector  $N = (N_1, \dots, N_{D-2})$  exists, where  $N_i = \partial_i f(x)$ , is that the sub-determinant  $g_{D-2}^{-1}$  of the contravariant metric tensor at the horizon ( $r_h$ ) must be equal to zero, namely,  $g_{D-2}^{-1}(r_h) = 0$ . As we assert that the metric determinant  $g_D$  is non-singular, and so is the contravariant metric determinant  $g_D^{-1}$ . From the equality  $g_{D-2}^{-1} = \mathcal{D}g_D^{-1}$ , one can know that the location of the horizon is given by the solutions of the following combination of equation and inequality:

$$g_{D-2}^{-1}(x) = 0, \quad \text{namely} \quad \mathcal{D}(x) = 0, \quad \text{and} \quad g_D(x) \neq 0. \quad (5)$$

Eq.(2.3) in Ref. 18 is a part of this combination of equation and inequality that determines the location of the horizon. For the metric component  $g_{\varphi\varphi}$  is nonzero at the event horizon ( $r_h$ ), this expression is equivalent to Eq.(4) in Ref. 16, namely,  $1/g^{tt}(r_h) = 0$ .

In general, we can rewrite function  $\mathcal{D}(x)$  around the point  $r_h$  as

$$\mathcal{D}(x) = (r - r_h)^\alpha H(x) \approx (r - r_h)^\alpha H(r_h), \quad (6)$$

here  $r$  is referred as to a radial coordinate in subspace,  $H(x)$  is an analytical function at the point  $r_h$ ,  $\alpha$  is the order of zeros of function  $\mathcal{D}(x)$ , or the order of poles of the sub-determinant  $g_{D-2}$ . In other words,  $\alpha$  is the number of duplicate roots of Eq.(5). For the flat space-time, no solution to Eq.(5) exists, so the index  $\alpha = 0$ ; For a non-extremal black holes, Eq.(5) has a single root  $r_h$ , so  $\alpha = 1$ ; In the extremal case, it has a double root  $r_h$ , so we have  $\alpha = 2$ , etc.

### 3 Deduction of momentum constraint and state density

Before calculating the state density of single particle for a given energy from the volume of phase space, we firstly derive the constraint on momentum space. We proceed with the Lagrange-Hamiltonian method rather than from the semi-classical approximation of Klein-Gordon equation.

The Lagrangian of a relativistic charged particle in the above background space-time (1) is given by

$$2\mathcal{L} = g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{\varphi\varphi}\dot{\varphi}^2 + g_{ij}\dot{x}^i\dot{x}^j + q(A_t\dot{t} + A_\varphi\dot{\varphi} + A_i\dot{x}^i). \quad (7)$$

Substituting the canonical conjugate momentum given by the following definition:

$$\begin{aligned} p_t &= \frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi} + qA_t = k_t + qA_t, \\ p_\varphi &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi} + qA_\varphi = k_\varphi + qA_\varphi, \\ p_i &= \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij}\dot{x}^j + qA_i = k_i + qA_i, \end{aligned}$$

into the covariant Hamiltonian defined by  $\mathcal{H} = p_t\dot{t} + p_\varphi\dot{\varphi} + p_i\dot{x}^i - \mathcal{L}$ , we get

$$2\mathcal{H} = g^{tt}(p_t - qA_t)^2 + 2g^{t\varphi}(p_t - qA_t)(p_\varphi - qA_\varphi) + g^{\varphi\varphi}(p_\varphi - qA_\varphi)^2 + g^{ij}(p_i - qA_i)(p_j - qA_j). \quad (8)$$

Hamilton constraint  $2\mathcal{H} = -\mu^2$  yields the constraint on momentum space:

$$\begin{aligned} g^{tt}k_t^2 + 2g^{t\varphi}k_tk_\varphi + g^{\varphi\varphi}k_\varphi^2 + g^{ij}k_ik_j + \mu^2 &= g^{ij}(p_i - qA_i)(p_j - qA_j) + \mu^2 \\ + g^{tt}(\omega + qA_t)^2 - 2g^{t\varphi}(\omega + qA_t)(m - qA_\varphi) + g^{\varphi\varphi}(m - qA_\varphi)^2 &= 0, \end{aligned} \quad (9)$$

where we have put  $p_t = -\omega$ ,  $p_\varphi = m$ , and  $-k_t = \omega + qA_t$ .

Let  $p_t = \partial_t \mathcal{S}$ ,  $p_\varphi = \partial_\varphi \mathcal{S}$ ,  $p_i = \partial_i S(x)$ , where  $\mathcal{S} = -\omega t + m\varphi + S(x)$ , we can derive Hamilton-Jacobi (H-J) equation which is a rather good semi-classical (W-K-B) approximation to Klein-Gordon equation for a complex scalar field  $\Psi = e^{i\mathcal{S}}$  in this geometry.

General speaking, the H-J equation only has well-meaning for a scalar field. However, we assume that constraint Eq.(9) works for particles obeying Maxwell- Boltzmann (M-B), Bose-Einstein (B-E) or Fermi-Dirac (F-D) statistics, and we will use it to calculate the density of single particle in the classical phase space. The reason is that the state density evaluated by the classical phase space method is a rather good approximation to degeneracy of discrete levels in quantum case, while the latter is, in general, very difficult to be dealt with.

Secondly, we evaluate the density of single particle state in the case that a quantum field has a vanishing angular velocity  $\Omega_0 = 0$ . (For case  $\Omega_0 \neq 0$ , see below). Momentum constraint of Eq.(9) can be recast into form

$$\frac{k_ik_j}{g_{ij}} + \frac{-g_{tt}}{-\mathcal{D}}[m - qA_\varphi + \frac{g_{t\varphi}}{g_{tt}}(\omega + qA_t)]^2 = \frac{1}{-g_{tt}}(\omega + qA_t)^2 - \mu^2, \quad (10)$$

On the one hand, for a given energy  $\omega$ , the hypersurface represented by Eq.(10) is a ellipsoid in  $(D - 1)$  dimensional momentum space supposed that it satisfies the following conditions:

$$g_{ij} > 0, \frac{-g_{tt}}{-\mathcal{D}} > 0, \frac{(\omega + qA_t)^2}{-g_{tt}} > \mu^2.$$

To prevent from appearing an infinite and imaginary state density, these conditions must be satisfied. Thus we need to restrict the system in the region that  $g_{tt} < 0$ . As the metric signature is taken as  $(-, +, \dots, +)$ , so we need  $g_{ij} > 0, -g_{tt} > 0$ , then the above conditions reduce to

$$-g_{tt} > 0, \quad -\mathcal{D} > 0, \quad \frac{(\omega + qA_t)^2}{-g_{tt}} > \mu^2. \quad (11)$$

The first condition and second one place restriction on the coordinate space, the third one gives the lower bound on the range of energy. If these three conditions are satisfied, then the volume of phase space is finite. Otherwise, the hyper-surface is noncompact, and the state density  $g(\omega)$  is divergent. The density of states for a given energy is given by taking differentiation of phase volume with respect to energy,  $g(\omega) = d\Gamma(\omega)/(d\omega)$ , where  $\Gamma(\omega)$  is the volume of  $2(D - 1)$  dimensional phase space for a given energy  $\omega$ :

$$\begin{aligned} \Gamma(\omega) &= \frac{1}{(2\pi)^{D-1}} \int d^{D-2}x d\varphi \int dm dk^{D-2} \\ &= \frac{1}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D+1}{2})} \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{-g_{tt}}} \left[ \frac{(\omega + qA_t)^2}{-g_{tt}} - \mu^2 \right]^{\frac{D-1}{2}}. \end{aligned} \quad (12)$$

It must be emphasized that a factor  $1/2$ , though not important to the final results, had been ignored in many literatures<sup>4,6,8,10,11,13,14,16</sup> in which the authors who used the radial wave number to calculate the free energy of a scalar field. The reason is that when one takes square roots of the radial wave number, he only takes a positive root and gets rid of a negative one. Physically, one discards the negative wave number; Mathematically, he gives up another leaf of paraboloid represented by the radial wave number. In the case of Minkowski space-time, the phase volume calculated by Eq.(12) is equal to the volume of the coordinate space times the volume of the momentum space being divided by a Planckian phase factor  $(2\pi)^3$ . Thus the total number of single particle state computed from Eq.(12) is correct.

On the other hand, for a given energy  $\omega$  and a fixed azimuthal angular momentum  $m$ , Eq.(9) represents a compact surface in  $(D - 2)$  dimensional momentum space provided it satisfies the following angular momentum-energy constraint condition:

$$g^{tt}(\omega + qA_t)^2 - 2g^{t\varphi}(\omega + qA_t)(m - qA_\varphi) + g^{\varphi\varphi}(m - qA_\varphi)^2 + \mu^2 = -g^{ij}k_ik_j \leq 0, \quad (13)$$

due to  $g_{ij} > 0, k_i^2 \geq 0$ . Thus the volume of  $(2D - 3)$  dimensional phase space is easily computed,

$$\Gamma(\omega, m) = \frac{1}{(2\pi)^{D-2}} \int d^{D-2}x d\varphi \int dk^{D-2} = \frac{1}{(4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} \int d^{D-2}x d\varphi \sqrt{g_{D-2}} \\ \times [-g^{tt}(\omega + qA_t)^2 + 2g^{t\varphi}(\omega + qA_t)(m - qA_\varphi) - g^{\varphi\varphi}(m - qA_\varphi)^2 - \mu^2]^{\frac{D-2}{2}}. \quad (14)$$

The state density or the number of modes for a given  $\omega$  and a fixed  $m$  in  $(2D - 3)$  dimensional space is given by  $g(\omega, m) = d\Gamma(\omega, m)/(d\omega)$ . Using condition (13) and carrying out the integration or summation with respect to  $m$ , namely,

$$\Gamma(\omega) = \frac{1}{2\pi} \int dm \Gamma(\omega, m) = \sum_m \Gamma(\omega, m),$$

we can get the same result of the total number of state for a given energy  $\omega$

$$g(\omega) = \frac{1}{2\pi} \int dm g(\omega, m) = \sum_m g(\omega, m).$$

Here and after we take the quantum number  $m$  as a continuous variable.

#### 4 Thermodynamical potential in the case ( $\Omega_0 = 0$ )

The volume of phase space in Eq.(12) and its corresponding state density  $g(\omega)$  are suitable to a quantum field that has a vanishing angular velocity but can have a potential  $\Phi_0$ . It is very convenient to use them in the case of static black holes and the flat space-time. We assume that a general relativistical bosonic, fermionic idea gas or non-interaction classical Boltzmann gas is in thermal equilibrium at temperature  $1/\beta$  in the background space-time (1). The free energy for three kinds of the usual statistics is given by

$$\beta F = \begin{cases} -\sum_m \int d\omega g(\omega, m) e^{-\beta(\omega - q\Phi_0)}, & (M - B), \\ \sum_m \int d\omega g(\omega, m) \ln(1 - e^{-\beta(\omega - q\Phi_0)}), & (B - E), \\ -\sum_m \int d\omega g(\omega, m) \ln(1 + e^{-\beta(\omega - q\Phi_0)}), & (F - D). \end{cases} \quad (15)$$

After carrying the integration by parts on the r.h.s in Eq.(15) and making a substitution  $\mathcal{E} = \omega + qA_t, \mathcal{B} = \Phi_0 + A_t$ , the above equation becomes

$$-F = \int d\omega \Gamma(\omega) \begin{cases} \frac{1}{e^{\beta(\omega - q\Phi_0)} - 1}, \\ \frac{1}{e^{\beta(\omega - q\Phi_0)} + 1} \end{cases} = \int d\mathcal{E} \Gamma(\mathcal{E}) \begin{cases} \frac{1}{e^{\beta(\mathcal{E} - q\mathcal{B})} - 1}, \\ \frac{1}{e^{\beta(\mathcal{E} - q\mathcal{B})} + 1} \end{cases} \begin{cases} (M - B), \\ (B - E), \\ (F - D). \end{cases} \quad (16)$$

The substitution  $\mathcal{E} = -k_t = -p_t + qA_t = \omega + qA_t$  corresponds to a gauge transformation which doesn't alter the volume of phase space, thus the density of state is invariant under such a gauge transformation. So we have  $\Gamma(\omega) = \Gamma(\mathcal{E})$ . One can always choose such a gauge  $\Phi_0$  that makes  $\mathcal{B} = \Phi_0 + A_t = 0$ . Substituting the total number of state (namely, phase volume)

$$\Gamma(\mathcal{E}) = \frac{1}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D+1}{2})} \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{-g_{tt}}} \left[ \frac{\mathcal{E}^2}{-g_{tt}} - \mu^2 \right]^{\frac{D-1}{2}}, \quad (17)$$

into the r.h.s of the second one in Eq.(16), we get the expression for the free energy

$$\begin{aligned} -F = & \frac{1}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D+1}{2})} \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{-g_{tt}}} \int_{\mu\sqrt{-g_{tt}}}^{\infty} d\mathcal{E} \\ & \times \left[ \frac{\mathcal{E}^2}{-g_{tt}} - \mu^2 \right]^{\frac{D-1}{2}} \begin{cases} e^{-\beta(\mathcal{E}-q\mathcal{B})}, & (M-B) \\ \frac{1}{e^{\beta(\mathcal{E}-q\mathcal{B})}-1}, & (B-E) \\ \frac{1}{e^{\beta(\mathcal{E}-q\mathcal{B})}+1}, & (F-D). \end{cases} \end{aligned} \quad (18)$$

If  $\mu\sqrt{-g_{tt}} > q\mathcal{B}$ , then  $\mathcal{E} > q\mathcal{B}$ , in this case there exists no superradiant mode for a quantum bosonic field. However, if  $\mu\sqrt{-g_{tt}} < q\mathcal{B}$ , then there is an energy interval  $\mu\sqrt{-g_{tt}} \leq \mathcal{E} < q\mathcal{B}$  corresponding to the superradiant mode ( $\omega < q\Phi_0$ ) as well as another interval  $\mathcal{E} > q\mathcal{B}$  corresponding to the non-superradiant mode ( $\omega > q\Phi_0$ ). It is well known that there is no superradiant effect for a fermionic field. Because it is somewhat complicated in the superradiant case, we here shall not cope with it. Let us suppose  $\mathcal{E} \geq \mu\sqrt{-g_{tt}} > q\mathcal{B} \geq 0$ , then after some calculation, we get the final results for the free energy in this case.

$$-F = 2\left(\frac{\mu}{2\pi}\right)^{D/2} \int d^{D-2}x d\varphi \frac{\sqrt{-g_D}}{(\beta\sqrt{-g_{tt}})^{D/2}} \begin{cases} e^{\beta q\mathcal{B}} K_{D/2}(\mu\beta\sqrt{-g_{tt}}), & (M-B), \\ \sum_{n=1}^{\infty} \frac{e^{n\beta q\mathcal{B}} K_{D/2}(n\mu\beta\sqrt{-g_{tt}})}{n^{D/2}}, & (B-E), \\ \sum_{n=1}^{\infty} \frac{e^{n\beta q\mathcal{B}} (-1)^{n+1} K_{D/2}(n\mu\beta\sqrt{-g_{tt}})}{n^{D/2}}, & (F-D). \end{cases} \quad (19)$$

Here  $K_{D/2}$  being the modified Bessel (or MacDonald) function of  $D/2$ -th order.

By using the asymptotic expression of  $D/2$ -th order MacDonald function  $K_{D/2}(z)$  at small  $z$ :

$$K_{D/2}(z) \approx \frac{2^{\frac{D-2}{2}} \Gamma(D/2)}{z^{D/2}}, \quad z \rightarrow 0$$

one can obtain the asymptotic behavior of the free energy in the high temperature approximation ( $\beta \rightarrow 0$ ) or in the zero mass case ( $\mu = 0$ )



$$\begin{aligned}
-F &\approx \frac{\Gamma(D/2)}{\pi^{D/2}} \int d^{D-2}x d\varphi \frac{\sqrt{-g_D}}{(\beta\sqrt{-g_{tt}})^D} \left\{ \begin{aligned} &e^{\beta q \mathcal{B}} \\ &\sum_{n=1}^{\infty} \frac{(e^{\beta q \mathcal{B}})^n}{n^D} \\ &-\sum_{n=1}^{\infty} \frac{(-e^{\beta q \mathcal{B}})^n}{n^D} \end{aligned} \right. \\
&= \frac{\Gamma(D/2)V_{D-1}}{\pi^{D/2}\beta^D} \left\{ \begin{aligned} &e^{\beta q \mathcal{B}}, & (M-B) \\ &\zeta_D(e^{\beta q \mathcal{B}}), & (B-E) \\ &-\zeta_D(-e^{\beta q \mathcal{B}}), & (F-D) \end{aligned} \right. \quad (20)
\end{aligned}$$

Here  $\zeta_D(s) = \sum_{n=1}^{\infty} \frac{s^n}{n^D}$  being Riemann zeta function.  $V_{D-1} = \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{(-g_{tt})^D}}$  is the volume of optical space<sup>15</sup> with metric determinant  $\bar{g}_D = g_D/(-g_{tt})^D$ . The metric of optical space is

$$d\bar{s}^2 = dt^2 + 2\frac{g_{t\varphi}}{g_{tt}}dt d\varphi + \frac{g_{\varphi\varphi}}{g_{tt}}d\varphi^2 + \frac{g_{ij}}{g_{tt}}dx^i dx^j. \quad (21)$$

If we choose such a gauge potential  $\Phi_0$  that makes  $\mathcal{B} = 0$ , and introduce a convenient statistical factor  $\mathcal{N}_D = 1, \zeta_D(1), -\zeta_D(-1)$ , for M-B, B-E, F-D statistics, respectively, then the free energy will be

$$-F \approx \mathcal{N}_D \frac{\Gamma(D/2)V_{D-1}}{\pi^{D/2}\beta^D}. \quad (22)$$

The free energy in Eq.(22) coincides with that calculated by the heat kernel expansion method in Ref. 15. In the case of a Dirac field, they only differs by a factor  $2^{D/2}$  which is the number of a Dirac spinor components in a  $D$ -dimensional space and isn't considered by us here. In Ref. 15, the free energy is derived by functional integral method and made a heat kernel expansion in the high temperature approximation in the static space-time. However, the start-point of ours is a general stationary axisymmetry space-time.

The entropy is given by

$$S = \beta^2 \frac{\partial F}{\partial \beta} \approx \mathcal{N}_D \frac{D\Gamma(D/2)V_{D-1}}{\pi^{D/2}\beta^{D-1}}. \quad (23)$$

The free energy in Eq.(22) and the entropy in Eq.(23) are proportional to the volume of optical space and depend on the dimensional number  $D$  of the considered space-time. Their dependence on the space-time is only related to the temporal component of the metric tensor. If the metric tensor  $g_{tt}$  is nonzero everywhere in the space-time, there exists no divergence. However, if  $g_{tt}$  vanishes at somewhere, then divergences appear there. Our results agree with that in Ref. 15 in the case of a four dimensional space-time. In the

case of a black hole, only after introducing a brick wall cutoff and subtracting minor terms, can the entropy be proportional to the "area" of the event horizon at the Hartle-Hawking temperature  $1/\beta_h = \kappa/(2\pi)$ . (This will be illustrated below.)

## 5 Thermodynamical potential in the case ( $\Omega_0 \neq 0$ )

However, in the case that a quantum field has a nonzero angular velocity  $\Omega_0$  in the stationary axisymmetry space-time, the matter is slightly different. Zhao and Gui<sup>19</sup> pointed out that "physical space" must be dragged by gravitational field with azimuthal angular velocity  $\Omega_H$ , and this is also noticed by Lee and Kim in Ref. 7-10 and other author<sup>16</sup>. A classical relativistic idea gas or a quantum field in thermal equilibrium at temperature  $1/\beta$  in this background must be dragged too. Therefore, it can be reasonable to assume that the quantum field or the classical particle is rotating with an azimuthal angular velocity  $\Omega_0(x)$  and has a potential  $\Phi_0(x)$ . For such a modified angular momentum-energy equilibrium ensemble<sup>14</sup> of states of the field, the thermodynamic potential of the system for particles with charge  $q$  and mass  $\mu$  is given by

$$\beta W = \begin{cases} -\sum_m \int d\omega g(\omega, m) e^{-\beta(\omega - m\Omega_0 - q\Phi_0)}, & (M - B), \\ \sum_m \int d\omega g(\omega, m) \ln(1 - e^{-\beta(\omega - m\Omega_0 - q\Phi_0)}), & (B - E), \\ -\sum_m \int d\omega g(\omega, m) \ln(1 + e^{-\beta(\omega - m\Omega_0 - q\Phi_0)}), & (F - D). \end{cases} \quad (24)$$

Let us define  $E - qB = \omega - m\Omega_0 - q\Phi_0$ ,  $B = \Phi_0 + A_t + \Omega_0 A_\varphi$ , then after carrying out the integration by parts in the r.h.s of Eq.(24), we obtain

$$-W = \int dE \Gamma(E) \begin{cases} e^{-\beta(E - qB)}, & (M - B), \\ \frac{1}{e^{\beta(E - qB)} - 1}, & (B - E), \\ \frac{1}{e^{\beta(E - qB)} + 1}, & (F - D). \end{cases} \quad (25)$$

The total number of states  $\Gamma(E)$  is obtained by substituting  $\omega + qA_t = E + \Omega_0(m - qA_\varphi)$  into Eq.(9) and reducing this equation to Eq.(28) (see below). Now it is expressed as

$$\Gamma(E) = \frac{1}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D+1}{2})} \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{-\tilde{g}_{tt}}} \left[ \frac{E^2}{-\tilde{g}_{tt}} - \mu^2 \right]^{\frac{D-1}{2}}, \quad (26)$$

here we have put  $\tilde{g}_{tt} = g_{tt} + 2g_{t\varphi}\Omega_0 + g_{\varphi\varphi}\Omega_0^2$ .

The finiteness of the state density is guaranteed by the following conditions

$$-\tilde{g}_{tt} > 0, \quad -\mathcal{D} > 0, \quad \frac{E^2}{-\tilde{g}_{tt}} > \mu^2 \quad (27)$$

which comes from the compactness of hypersurface determined by

$$\frac{k_i k_j}{g_{ij}} + \frac{-\tilde{g}_{tt}}{-\mathcal{D}} [m - qA_\varphi + \frac{g_{t\varphi} + g_{\varphi\varphi}\Omega_0}{\tilde{g}_{tt}} E]^2 = \frac{E^2}{-\tilde{g}_{tt}} - \mu^2. \quad (28)$$

To preserve the state density real and finite, we must restrict the system in the region that satisfies  $-\tilde{g}_{tt} > 0$  because we want  $-\mathcal{D} > 0$  as before. This imposes restrictions on the angular velocity in the region that  $\Omega - \sqrt{-\mathcal{D}}/g_{\varphi\varphi} < \Omega_0 < \Omega + \sqrt{-\mathcal{D}}/g_{\varphi\varphi}$ , where  $\Omega = -g_{t\varphi}/g_{\varphi\varphi}$ . Suppose that  $E \geq \mu\sqrt{-\tilde{g}_{tt}} > qB \geq 0$ , namely, we only consider the case that the non-superradiant mode exists for a scalar field, as the calculation is somewhat complicated in the superradiant case. Substituting the total number of single particle state  $\Gamma(E)$  into the thermodynamical potential, we have

$$\begin{aligned} -W = & \frac{1}{(4\pi)^{\frac{D-1}{2}} \Gamma(\frac{D+1}{2})} \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{-\tilde{g}_{tt}}} \int_{\mu\sqrt{-\tilde{g}_{tt}}}^{\infty} dE \\ & \times \left[ \frac{E^2}{-\tilde{g}_{tt}} - \mu^2 \right]^{\frac{D-1}{2}} \begin{cases} e^{-\beta(E-qB)}, & (M-B) \\ \frac{1}{e^{\beta(E-qB)}-1}, & (B-E) \\ \frac{1}{e^{\beta(E-qB)}+1}, & (F-D). \end{cases} \end{aligned} \quad (29)$$

The thermodynamical potential for Bose-Einstein statistics coincides with Eq.(19) in Ref. 9 for a quantum scalar field in a four dimensional Kerr-Newman black hole geometry. Under our assumption that  $E \geq \mu\sqrt{-\tilde{g}_{tt}} > qB \geq 0$ , we can always select such a gauge potential  $\Phi_0$  that makes  $B = 0$  without violating the above assumption. After carrying out the integration with respect to  $E$ , we arrive at results

$$-W = 2\left(\frac{\mu}{2\pi}\right)^{D/2} \int d^{D-2}x d\varphi \frac{\sqrt{-g_D}}{(\beta\sqrt{-\tilde{g}_{tt}})^{D/2}} \begin{cases} K_{D/2}(\mu\beta\sqrt{-\tilde{g}_{tt}}), & (M-B), \\ \sum_{n=1}^{\infty} \frac{K_{D/2}(n\mu\beta\sqrt{-\tilde{g}_{tt}})}{n^{D/2}}, & (B-E), \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} K_{D/2}(n\mu\beta\sqrt{-\tilde{g}_{tt}})}{n^{D/2}}, & (F-D). \end{cases} \quad (30)$$

In the high temperature approximation ( $\beta \rightarrow 0$ ) or in the zero mass case ( $\mu = 0$ ), the thermodynamical potential has asymptotic behavior:

$$-W \approx \frac{\Gamma(D/2)}{\pi^{D/2}} \int d^{D-2}x d\varphi \frac{\sqrt{-g_D}}{(\beta\sqrt{-\tilde{g}_{tt}})^D} \mathcal{N}_D = \mathcal{N}_D \frac{\Gamma(D/2) \tilde{V}_{D-1}}{\pi^{D/2} \beta^D}. \quad (31)$$

Here  $\tilde{V}_{D-1} = \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{(-\tilde{g}_{tt})^D}}$  is the volume of the dragged optical space with determinant  $\tilde{g}_D = g_D/(-\tilde{g}_{tt})^D$ . The metric of the dragged optical space is

$$d\tilde{s}^2 = dt^2 + 2\frac{g_{t\varphi} + g_{\varphi\varphi}\Omega_0}{\tilde{g}_{tt}}dtd\varphi + \frac{g_{\varphi\varphi}}{\tilde{g}_{tt}}d\varphi^2 + \frac{g_{ij}}{\tilde{g}_{tt}}dx^i dx^j. \quad (32)$$

The entropy in the high temperature approximation is given by

$$S = \beta^2 \frac{\partial W}{\partial \beta} \approx \mathcal{N}_D \frac{D\Gamma(D/2)\tilde{V}_{D-1}}{\pi^{D/2}\beta^{D-1}}. \quad (33)$$

The thermodynamical potential and its corresponding entropy are proportional to the volume of the dragged optical space in the high temperature approximation. Apparently, they depend on the temperature and the dimensional number of the considered space-time as well as the temporal component of the dragged metric. Their divergences count on the property of the dragged metric tensor  $\tilde{g}_{tt}$ . No divergence will appear when  $\tilde{g}_{tt}$  is nonzero everywhere. When the angular velocity  $\Omega_0$  vanishes, they will degenerate to the case considered in the last section.

Using the asymptotic expression of  $D/2$ -th order MacDonald function  $K_{D/2}(z)$  at large  $z$  and taking only its zero order approximation

$$K_{D/2}(z) \approx \sqrt{\frac{\pi}{2z}}e^{-z}, \quad z \rightarrow \infty$$

we can obtain the asymptotic behavior of the thermodynamical potential in the low temperature approximation ( $\beta \rightarrow \infty$ ):

$$\begin{aligned} -W &\approx \left(\frac{\mu}{2\pi}\right)^{\frac{D-1}{2}} \int d^{D-2}x d\varphi \frac{\sqrt{-g_D}}{(\beta\sqrt{-\tilde{g}_{tt}})^{\frac{D+1}{2}}} \left\{ \begin{aligned} &e^{-\beta\mu\sqrt{-\tilde{g}_{tt}}} \\ &\sum_{n=1}^{\infty} \frac{e^{-n\beta\mu\sqrt{-\tilde{g}_{tt}}}}{n^{\frac{D+1}{2}}} \\ &\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{-n\beta\mu\sqrt{-\tilde{g}_{tt}}}}{n^{\frac{D+1}{2}}} \end{aligned} \right. \\ &= \left(\frac{\mu}{2\pi}\right)^{\frac{D-1}{2}} \int d^{D-2}x d\varphi \frac{\sqrt{-g_D}}{(\beta\sqrt{-\tilde{g}_{tt}})^{\frac{D+1}{2}}} \left\{ \begin{aligned} &e^{-\beta\mu\sqrt{-\tilde{g}_{tt}}}, & (M-B) \\ &\zeta_{\frac{D+1}{2}}[e^{-\beta\mu\sqrt{-\tilde{g}_{tt}}}], & (B-E) \\ &-\zeta_{\frac{D+1}{2}}[-e^{-\beta\mu\sqrt{-\tilde{g}_{tt}}}], & (F-D) \end{aligned} \right. \\ &\rightarrow 0, \quad \text{when } \beta \rightarrow \infty. \end{aligned} \quad (34)$$

In the low temperature approximation, the thermodynamical potential and the entropy  $S = \beta^2 \frac{\partial W}{\partial \beta}$  all exponentially tend to become zero suppose that  $\tilde{g}_{tt}$  remains finite at every point in the space-time. They will be divergent at the point where  $\tilde{g}_{tt}$  vanishes. This is physically reasonable and is consistent with the third law of the usual thermodynamics.

## 6 Discussion: horizon and divergence

We have stressed that whether the thermodynamical potential  $W$  (or the free energy  $F$ ) diverges or not apparently depends upon whether the metric tensor  $\tilde{g}_{tt}$  (or  $g_{tt}$ ) vanishes or not. The divergence appears if and only if  $\tilde{g}_{tt}$  is equal to zero. Although there exists a event horizon on a black hole background, however if  $\tilde{g}_{tt}$  is nonzero at this horizon, then no divergence will appear. Thus a necessary and sufficient condition that the divergence appears at the horizon is that the dragged metric tensor  $\tilde{g}_{tt}$  is equal to zero at this horizon. Because  $\tilde{g}_{tt}$  degenerates to  $g_{tt}$  when the angular velocity  $\Omega_0$  is zero, so we only need to study the general case, namely  $\Omega_0 \neq 0$ .

Suppose  $r_c$  is the  $\rho$ -fold root of equation:  $\tilde{g}_{tt} = 0$ , then we can recast  $\tilde{g}_{tt}$  around the point  $r_c$  into the form

$$\tilde{g}_{tt} = (r - r_c)^\rho G(x) \approx (r - r_c)^\rho G(r_c), \quad G(r_c) = \frac{1}{\rho!} \frac{d^\rho}{dr^\rho} \tilde{g}_{tt}(r_c) = \frac{1}{\rho!} \tilde{g}_{tt}^{(\rho)}(r_c), \quad (35)$$

here  $G(x)$  being an analytical function at the point  $r_c$ , the  $(D-2)$ -th coordinate  $x^{D-2} = r$  is a "radial" coordinate. In the first one of the above-head equation, the second expression is obtained by taking the lowest order approximation.

Actually the point  $r_c$  is located at the light velocity surface<sup>5,7-10</sup>. In the case  $\Omega_0 = 0$ , the surface such that  $g_{tt} = 0$  is the infinite red-shift surface. Apparently, the location of the light velocity surface depends upon the choice of the angular velocity  $\Omega_0$ . Further, let us assume<sup>1</sup> that the horizon is on the light velocity surface, namely, the location of the horizon satisfies equation  $\tilde{g}_{tt}(r_h) = 0$ . Near the horizon  $r_h$ , the dragged metric tensor  $\tilde{g}_{tt}$  tends to become zero.

From the asymptotic expression of  $D/2$ -th order MacDonald function  $K_{D/2}(z)$  at small  $z$ , one can know that the thermodynamical potential near the horizon has the same behavior as it does in the high temperature approximation or in the zero mass case:

$$-W \approx \frac{\Gamma(D/2)}{\pi^{D/2} \beta^D} \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{(-\tilde{g}_{tt})^D}} \mathcal{N}_D = \mathcal{N}_D \frac{\Gamma(D/2) \tilde{V}_{D-1}}{\pi^{D/2} \beta^D}. \quad (36)$$

Substituting the lowest approximation of the metric tensor  $\tilde{g}_{tt}$  into the expression of the volume of the dragged optical space, the leading behavior of  $\tilde{V}_{D-1}$  near the horizon is given by

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<sup>1</sup>In fact, this assumption is reasonable physically in the case of a black hole.

$$\begin{aligned}
\tilde{V}_{D-1} &= \int d^{D-2}x d\varphi \sqrt{\frac{-g_D}{(-\tilde{g}_{tt})^D}} \\
&\approx \int d^{D-3}x d\varphi \int_{r_h+\epsilon}^L dr (r-r_h)^{-D\rho/2} \sqrt{\frac{-g_D}{(-G)^D}}(r_h) \\
&\approx \frac{2}{D\rho-2} \epsilon^{1-D\rho/2} \int d^{D-3}x d\varphi \sqrt{\frac{-g_D}{(-G)^D}}(r_h).
\end{aligned} \tag{37}$$

Here we introduce a small cut-off  $\epsilon$  and another cut-off  $L \gg r_h$  to remove the infra-red divergence and the U-V divergence, respectively. The leading behavior of the entropy near the horizon is given by

$$S \approx \mathcal{N}_D \frac{2D\Gamma(D/2)}{(D\rho-2)\pi^{D/2}\beta^{D-1}} \epsilon^{1-D\rho/2} \mathcal{F}(r_h), \tag{38}$$

here  $\mathcal{F}(r_h) = \int d^{D-3}x d\varphi \sqrt{\frac{-g_D}{(-G)^D}}(r_h)$  is proportional to the "area" of the event horizon.

The leading entropy of an idea gas obeying three kinds of the usual statistics diverges in  $\epsilon^{1-D\rho/2}$  as the system approaches the horizon of a black hole if and only if the location of the horizon is located at the light velocity surface. Under such a circumstance, the leading behavior of the entropy at temperature  $1/\beta = \kappa/(2\pi)$  is proportional to the horizon "area", but diverges as the brick wall cut-off  $\epsilon$  goes to zero. The divergence depends on the dimensional number  $D$  of the space-time as well as the degeneracy  $\rho$  of the horizon. The fundamental reason of the divergence is that the volume of the dragged optical space tends to become infinite near the horizon which results in that the density of states for a given energy  $E$  diverges as the system approaches the horizon.

However, although there doesn't exist a horizon in a non-black-hole space-time, the divergence will also appear when the system approaches the light velocity surface. Thus our conclusion is that the divergence has no direct relation to the horizon and it is only determined by the equation of the light velocity surface. A necessary condition that the divergence appears is that the horizon is located at the light velocity surface. In the following section, we will use some concrete examples to illustrate this viewpoint.

## 7 Examples

In this section, on the one hand, we will give some concrete examples to discuss the divergence problem, on the other hand, we will determine the location of the event horizon and its surface gravity in a given black hole geometry.

**Example A: Minkowski space-time ( $\alpha = 0$ )**

In the four dimensional flat geometry,  $g_{tt} = -1, A_t = 0, A_\varphi = 0$ , one can select a gauge that satisfies  $\mathcal{B} = \Phi_0 = 0$ . If a quantum field has a vanishing angular velocity  $\Omega_0 = 0$ , then the total number of states and the free energy are given by

$$\Gamma(\mathcal{E}) = \frac{1}{(4\pi)^{3/2}\Gamma(5/2)} \int d^3x \sqrt{-g_4} [\mathcal{E}^2 - \mu^2]^{3/2} = \frac{V_3}{6\pi^2} [\mathcal{E}^2 - \mu^2]^{3/2}, \quad (39)$$

$$-F = 2V_3 \left( \frac{\mu}{2\pi\beta} \right)^2 \begin{cases} K_2(\mu\beta), & (M - B), \\ \sum_{n=1}^{\infty} \frac{K_2(n\mu\beta)}{n^2}, & (B - E), \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} K_2(n\mu\beta)}{n^2}, & (F - D). \end{cases} \quad (40)$$

Here the volume of the optical space  $V_3 = \int_{\text{box}} d^3x \sqrt{-g_4}$  is equal to the volume of the box that confines the idea gas being considered.

In the low temperature approximation, the free energy and the entropy all tend to become zero in  $e^{-\beta\mu}$ . In the high temperature approximation, the asymptotic behavior of the entropy is given by

$$S(\Omega_0 = 0) \approx \mathcal{N}_4 \frac{4V_3}{\pi^2 \beta^3}, \quad (41)$$

here for convenience, the statistical factor is introduced  $\mathcal{N}_4 = 1, \zeta_4(1), -\zeta_4(-1)$ , for M-B, B-E, F-D statistics, respectively. The Riemann zeta constants are all known  $\zeta_4(1) = \pi^4/90, \zeta_4(-1) = 7/8\zeta_4(1) = 7\pi^4/720$ .

In the spherical coordinates frame, the Minkowski metric is given by  $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ , the metric determinant is  $g_4 = -r^4 \sin^2\theta$ , the volume of optical space is equal to that of the box, namely,  $V_3 = 4\pi \int_0^R r^2 dr = \frac{4\pi}{3} R^3$ .

No horizon exists in the flat space-time for there is no solution satisfying  $\mathcal{D} = -r^2 \sin^2\theta = 0, g_4 = -r^4 \sin^2\theta \neq 0$ , so the indices  $\alpha = 0, \rho = 0$ , the latter due to  $g_{tt} = -1 \neq 0$ . To make states density and entropy finite, one must restrict the size of 3-dimensional sphere, and use a box to confine the propagation of a quantum field. The box acts as imposing a boundary condition on the quantum field. This will result in quantization of energy and discretization of phase space volume as well as state density.

However, when a quantum field has a non-zero angular velocity  $\Omega_0 \neq 0$ , the dragged metric tensor is  $\tilde{g}_{tt} = -1 + r^2 \sin^2\theta \Omega_0^2$ . The thermodynamical potential is given by

$$-W = 2\left(\frac{\mu}{2\pi\beta^2}\right)^2 \int dr d\theta d\varphi \frac{r^2 \sin \theta}{-\tilde{g}_{tt}} \begin{cases} K_2(\mu\beta\sqrt{-\tilde{g}_{tt}}), & (M - B), \\ \sum_{n=1}^{\infty} \frac{K_2(n\mu\beta\sqrt{-\tilde{g}_{tt}})}{n^2}, & (B - E), \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} K_2(n\mu\beta\sqrt{-\tilde{g}_{tt}})}{n^2}, & (F - D). \end{cases} \quad (42)$$

The entropy at small  $\beta\mu\sqrt{-\tilde{g}_{tt}}$  behaves like

$$S(\Omega_0 \neq 0) \approx \mathcal{N}_4 \frac{4\tilde{V}_3}{\pi^2\beta^3}, \quad (43)$$

here the volume of the dragged optical space  $\tilde{V}_3 = 2\pi \int dr d\theta r^2 \sin \theta (1 - r^2 \sin^2 \theta \Omega_0^2)^{-2}$ .

The entropy diverges as the angular velocity  $|\Omega_0| \rightarrow (r \sin \theta)^{-1}$ . To preserve the volume of the dragged optical space and the entropy finite and real, one must restrict the velocity  $|\Omega_0| < (r \sin \theta)^{-1}$ . Otherwise the entropy will diverge or become imaginary. For a finite  $\Omega$ , the light velocity surface is located at the surface that  $r \sin \theta = \pm \Omega_0^{-1}$ , then the index  $\rho = 1$ . Under the condition  $|\Omega_0 r \sin \theta| < 1$ , the volume of the dragged optical space is

$$\begin{aligned} \tilde{V}_3 &= 2\pi \sum_{k=0}^{\infty} (k+1) \Omega_0^{2k} \int_0^R r^{2k+2} dr \int_0^\pi \sin^{2k+1} \theta d\theta \\ &= 4\pi R^3 \left\{ 1 + \sum_{k=1}^{\infty} \frac{[(k+1)!]^2}{(2k+2)!(k+3/2)} (2\Omega_0 R)^{2k} \right\}. \end{aligned} \quad (44)$$

The condition that the power series in the above-head expression converges is  $|\Omega_0 R| < 1$ . When the angular velocity  $\Omega_0 = 0$ , the dragged volume is equal to that of the sphere  $\tilde{V}_3^0 = 4\pi R^3/3$ . When  $\Omega_0 \neq 0$ , its zero-th order approximation is also equal to the sphere volume.

### Example B: Four dimensional static black hole ( $\alpha = 1; 2$ )

Next let us consider the Reissner-Nordstrom black hole, the metric and electro-magnetical potential are

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (45)$$

$$A = -\frac{Q}{r} dt, \quad \Delta = r^2 - 2Mr + Q^2 \quad (46)$$

The metric determinant is  $g_4 = -r^4 \sin^2 \theta$ . Conditions  $\mathcal{D} = -\Delta \sin^2 \theta = 0$ ,  $g_4 = r^4 \sin^2 \theta \neq 0$  can be satisfied by the horizon surface equation  $\Delta = 0$ . In the non-extremal case ( $M^2 \neq Q^2$ ), the metric has two horizons at  $r_{\pm} = M \pm (M^2 - Q^2)^{1/2}$ . Let  $r_h = r_{\pm}$ , then the horizon function can be rewritten as  $\Delta = (r - r_h)[r - r_h + 2(r_h - M)] \approx 2(r - r_h)(r_h - M)$ . So we have the index  $\alpha = 1$  and function  $H(r_h) = -2(r_h - M) \sin^2 \theta$ .



To preserve the state density real and finite, we should restrict the system in the region that  $\Delta > 0$  due to  $-g_{tt} > 0$ . Suppose that the angular velocity is zero,  $\Omega_0 = 0$ , and choose such a potential  $\Phi_0 = -A_t = Q/r$  that makes the chemical potential  $\mathcal{B} = 0$ , then the light velocity surface coincides the the horizon due to  $g_{tt} = -\Delta/r^2 = 0$ . So we have the index  $\rho = 1$  and constant  $G(r_h) = -2(r_h - M)/r_h^2 = -2\kappa_h$ . The leading term of the volume of the optical space is given by

$$V_3 = 4\pi \int dr \frac{r^6}{\Delta^2} \approx \int_{r_h+\epsilon}^L dr \frac{\pi r_h^6}{(r-r_h)^2(r_h-M)^2} \approx \frac{\pi r_h^6}{\epsilon(r_h-M)^2} = \frac{\mathcal{A}_h}{4\epsilon\kappa_h^2}, \quad (47)$$

here  $\mathcal{A}_h = 4\pi r_h^2$  is the area of the horizon with the surface gravity being  $\kappa_h = (r_h - M)/r_h^2$ . In terms of the proper distance cut-off from the horizon  $r_h$  to  $r_h + \epsilon$ :

$$\delta = \int_{r_h}^{r_h+\epsilon} \Delta^{-1/2} r dr \approx \int_{r_h}^{r_h+\epsilon} dr / \sqrt{2\kappa_h(r-r_h)} = \sqrt{2\epsilon/\kappa_h},$$

the volume of the optical space is rewritten as  $V_3 = \mathcal{A}_h/(2\kappa_h^3\delta^2)$ . The leading behavior of the entropy near the horizon is:

$$S(\Omega_0 = 0) \approx \mathcal{N}_4 \frac{2\mathcal{A}_h}{\pi^2(\beta\kappa_h)^3\delta^2}. \quad (48)$$

At the Hartle-Hawking temperature  $\beta = \kappa_h/(2\pi)$ , the entropy of an idea gas is proportional to the horizon area  $\mathcal{A}_h$  and diverges in  $\delta^{-2}$ :

$$S(\Omega_0 = 0) \approx \mathcal{N}_4 \frac{\mathcal{A}_h}{4\pi^5\delta^2}. \quad (49)$$

In the case of the extremal Reissner-Nordstrom black hole ( $M^2 = Q^2$ ), the horizon location  $r_h = M$  is the double root of equation  $\Delta = (r - M)^2 = 0$ , so the indices  $\alpha = \rho = 2$ . As the function  $\mathcal{D} = -(r - M)^2 \sin^2 \theta$ ,  $g_{tt} = -(r - M)^2/r^2 \approx -(r - M)^2 M^{-2}$ , so we have constant  $G(r_h) = -M^{-2}$  and function  $H(r_h) = -\sin^2 \theta$ . The leading term of the volume of the optical space near the horizon is given by

$$V_3 = \int 4\pi dr \frac{r^6}{\Delta^2} \approx \int_{M+\epsilon}^L dr \frac{4\pi M^6}{(r-M)^4} \approx \frac{4\pi M^6}{3\epsilon^3}. \quad (50)$$

Near the horizon, the entropy diverges cubically (in  $\epsilon^{-3}$ ):

$$S(\Omega_0 = 0) \approx \mathcal{N}_4 \frac{16M^6}{3\pi(\beta\epsilon)^3}. \quad (51)$$

The entropy of a quantum field in the extremal Reissner-Nordstrom black hole is proportional to the horizon area  $\mathcal{A}_e = 4\pi M^2$  only when the temperature of the system  $\beta \sim M^{4/3}$ .

The reason why the thermodynamics of the extremal black hole is ill-defined, however, is still unclear<sup>15</sup>.

Example C: Four dimensional stationary axisymmetry black hole ( $\alpha = 1$ )

The third example in which we have an interest is the Kerr-Newman black hole. In the Boyer-Lindquist coordinates, the metric and the electro-magnetical potential of the Kerr-Newman black hole takes the form

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - 2\frac{r^2 + a^2 - \Delta}{\Sigma} a \sin^2 \theta dt d\varphi \\ + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\varphi^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right), \quad (52)$$

$$A = -\frac{Qr}{\Sigma} (dt - a \sin^2 \theta d\varphi) \quad (53)$$

with the event horizon function  $\Delta = r^2 - 2Mr + Q^2 + a^2$ ,  $\Sigma = r^2 + a^2 \cos^2 \theta$ ,  $\mathcal{D} = -\Delta \sin^2 \theta$ , and the metric determinant  $g_4 = -(\Sigma \sin \theta)^2$ . The location of the horizon  $r_h = r_{\pm} = M \pm (M^2 - Q^2 - a^2)^{1/2}$  satisfies conditions  $\mathcal{D} = 0$ ,  $g_4 \neq 0$  by equation  $\Delta = 0$ . For the non-extremal case, the index is equal to one ( $\alpha = 1$ ), and function  $H(r_h) = -2(r_h - M) \sin^2 \theta$ .

The entropy of a quantum scalar field in the non-extreme Kerr-Newman black hole had been considered by many authors<sup>7-9,16</sup> in the case that the scalar field is co-rotating with the black hole, namely the angular velocity is a constant  $\Omega_h = a/(r_h^2 + a^2)$ , and the potential  $\Phi_h = Qr_h/(r_h^2 + a^2)$ , thus the chemical potential tends to become zero near the horizon. Other than this choice and the trial choice  $\Omega_0 = 0$ , however, we choose a local angular velocity and a local potential as:

$$\Omega_0 = \Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}, \\ \Phi_0 = -(A_t + \Omega A_{\varphi}) = \frac{Qr(r^2 + a^2)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}. \quad (54)$$

The reason why we choose a local velocity and a local potential is that the light velocity surface depends upon the choice of the angular velocity. For instance, in the case of  $\Omega_0 = 0$ , the points satisfying  $\tilde{g}_{tt} = 0$  are on the stationary limit surface. The local velocity and potential on the horizon are  $\Omega_0 = \Omega_h$ ,  $\Phi_0 = \Phi_h$ , respectively. They tends to become zero at infinity. Under such a choice, the chemical potential is always equal to zero ( $B = 0$ ), and the dragged metric tensor becomes:

$$\tilde{g}_{tt} = g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2 = 1/g^{tt} = -\Delta\Sigma/[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta].$$

The light velocity surface coincides with the horizon under our choice because the equation  $\tilde{g}_{tt} = 1/g^{tt} = 0$  can be satisfied by  $\Delta = 0$ . Thus the index  $\rho = 1$ . To prevent from the presence of infinite and imaginary state density, one must restrict the system satisfying inequalities:  $(r^2 + a^2)^2 > \Delta a^2 \sin^2 \theta > 0$ . Also, it places a lower bound and an upper bound on the angular velocity  $\Omega_0$ :  $\Omega - \sqrt{-\mathcal{D}g_{\varphi\varphi}^{-2}} < \Omega_0 < \Omega + \sqrt{-\mathcal{D}g_{\varphi\varphi}^{-2}}$ . If only  $\Delta > 0$ , then the choice of the velocity  $\Omega_0 = \Omega$  certainly satisfies this restrictions.

The leading term of the volume of the dragged optical space is given by

$$\begin{aligned}\tilde{V}_3 &= \int_0^\pi d\theta d\varphi \int_{r_h+\epsilon}^L dr \sqrt{-g_4} \left( \frac{g_{\varphi\varphi}}{-\mathcal{D}} \right)^2 = 2\pi \int_0^\pi d\theta \int_{r_h+\epsilon}^L dr \frac{\sin \theta}{\Sigma \Delta^2} [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta]^2 \\ &\approx \int_0^\pi \frac{\sin \theta d\theta}{r_h^2 + a^2 \cos^2 \theta} \int_{r_h+\epsilon}^L dr \frac{\pi(r_h^2 + a^2)^4}{2(r_h - M)^2(r - r_h)^2} \approx \frac{\mathcal{A}_h}{4\epsilon\kappa_h^2} \times \frac{r_h^2 + a^2}{ar_h} \arctan \frac{a}{r_h}.\end{aligned}$$

The proper distance from  $r_h$  to  $r_h + \epsilon$  is a function of  $\theta$ :

$$\delta = \int_{r_h}^{r_h+\epsilon} \sqrt{\frac{\Sigma}{\Delta}} dr \approx \int_{r_h}^{r_h+\epsilon} dr \sqrt{\frac{r_h^2 + a^2 \cos^2 \theta}{2(r_h - M)(r - r_h)}} \approx \sqrt{\frac{2\epsilon(r_h^2 + a^2 \cos^2 \theta)}{r_h - M}}.$$

In terms of the proper distance cut-off  $\delta$ , the dragged optical volume can be rewritten as

$$\tilde{V}_3 \approx \frac{\mathcal{A}_h}{2\kappa_h^3 \delta^2} \times \frac{r_h^2 + a^2 \cos^2 \theta}{ar_h} \arctan \frac{a}{r_h} \approx \frac{\mathcal{A}_h}{2\kappa_h^3 \delta^2}, \quad (a \ll r_h).$$

Here  $\mathcal{A}_h = 4\pi(r_h^2 + a^2)$  is the area of the event horizon, and  $\kappa_h = (r_h - M)/((r_h^2 + a^2))$  is the surface gravity. In the second approximation, we have taken the slow rotating limit. The leading behavior of the entropy near the horizon is:

$$S \approx \mathcal{N}_4 \frac{2\mathcal{A}_h}{\pi^2 (\beta\kappa_h)^3 \delta^2}. \quad (55)$$

If we take  $1/\beta$  as the Hartle-Hawking temperature  $\kappa_h/(2\pi)$ , the entropy of an idea gas near the horizon is proportional to the horizon area  $\mathcal{A}_h$  and diverges in  $\delta^{-2}$  as  $\delta \rightarrow 0$ :

$$S \approx \mathcal{N}_4 \frac{\mathcal{A}_h}{4\pi^5 \delta^2}. \quad (56)$$

This leading behavior of the entropy of quantum fields near the horizon is a general form in the black hole background. It is proportional to the horizon area but diverges as the system approaches to the horizon. The reason of the divergence is due to the infinite state density for a given energy near the horizon. This agrees with the conclusions in Ref. 8 and 9.

As a brief summary, we has used the first example to demonstrate that although there exists no horizon in the flat space-time, if a quantum field has a vanishing angular velocity,

one must also impose a box on the system to prevent from the presence of infinite state density. In the case that a quantum field has a non-vanishing angular velocity, the entropy also diverges as the system approaches to the light velocity surface. To guarantee the state density finite and real, the angular velocity must be restricted in a certain region.

The second example has been used to show that the thermodynamics of the extremal black hole is very different from that of the non-extremal black hole. In the third one, we have chosen a local angular velocity and a local potential other than the popular uniform velocity. Both examples have shown that the leading behavior of the entropy of a relativistical idea gas near the horizon is proportional to the horizon area and diverges as the system approaches the horizon provided that the horizon is on the light velocity surface.

All examples have illustrated that the entropy of a quantum field is proportional to the volume of the optical space or that of the dragged optical space. In the case of a black hole, it is proportional to the horizon area only after introducing a brick wall cut-off. In the four dimensional black holes, the leading term of the entropy has a common character.

Other cases can also be considered. In a lower than four dimensional space-time, we have assumed that the statistics are the usual ones. However, this may be problematical. In 2+1 dimensional planar system, anyons obeys a novel fractional statistics<sup>20</sup>, neither the common Bose-Einstein statistics nor the well-known Fermi-Dirac statistics. Theoretically, fractional quantum Hall effect probably be interpreted by anyonic statistics<sup>21</sup>.

## 8 Conclusion

To summarize, a general framework of general relativistical thermodynamics for three kinds of the usual statistics has been done in a  $D$ -dimensional stationary axisymmetry space-time. We start from calculating the density of single particle by the classical phase space method. The density of single state is invariant under a gauge transformation, however, it is suffered by the dragging of the angular velocity. To proceed, we assume that it is effective in an arbitrary dimensional space-time for a relativistical idea gas obeying the usual Maxwell-Boltzmann, Bose-Einstein or Fermi-Dirac statistics. A particular needed notice is that this assumption is probably invalid in a space-time with its dimensional number lower than four. In a space-time higher than the usual four dimension, no such problem exists.

Thermodynamical quantities such as the free energy or the thermodynamical potential and the entropy of a quantum field are evaluated. Exact analytical expressions for the free energy or the thermodynamical potential are in terms of the modified Bessel functions. In the

high temperature approximation, the statistical entropy of a quantum field is proportional to the volume of optical space in the case of a vanishing angular velocity of the quantum field. This conclusion agrees with the results that the entropy of a quantum field in a static space-time background is obtained by heat kernel expansion in the same approximation in Ref. 15. In the case that a quantum has a non-vanishing angular velocity, the entropy is proportional to the volume of the dragged optical space. In the low temperature approximation, the entropy tends to become zero exponentially.

In general, the entropy of a quantum field in a  $D$ -dimensional space-time depends upon the temporal component of the metric tensor  $g_{tt}$  or the dragged metric tensor  $\tilde{g}_{tt}$  only, as well as the dimensional number  $D$ . If the dragged metric tensor  $\tilde{g}_{tt}$  doesn't vanish at every point of the space-time being considered, then the entropy is finite provided the system is confined by a box. In the case of a black hole, the leading term of the entropy near the horizon is proportional to the horizon "area" only when the horizon is located at the light velocity surface. The presence of horizon has no direct relation to the divergence of entropy, but it introduces an additional brick wall cut-off in place of the restrictions of a box. Although there exists a horizon of the black hole, the behavior of the entropy near the horizon will also be finite if the light velocity surface doesn't coincide with the horizon. The divergence near the horizon has a definite relation to the dimensional number of a space-time.

As examples, we have discussed the four dimensional entropy of flat space-time and that of black holes. The results agree with the already-known results in the literatures. Using our general formation, one can compare the behaviors among the entropies in different dimensional space-times. It might not be a toy of everything, however, we wish it would work at least in a higher dimensional space-time. In the case of a bosonic field, we don't consider such things as the contribution to the entropy from the superradiant modes and a probable existing phenomena of the well-known Bose-Einstein condensation here. In the process of our calculation of the free energy or the thermodynamical potential, we have only made a small fugacity expansion also. however, we expect to discuss them in other places.

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